ON SYMMETRIC OCTAHEDRON AND HEXAHEDRON IN SPHERICAL SPACE

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Abstract. In the present report closed integral formulae for the volumes of symmetric spherical octahedron and hexahedron are established. Trigonometrical identities between lengths of edges and dihedral angles (Sine-Tangent Rules) are obtained. This gives the possibility to express the lengths in terms of angles. Then the Schläfli formula is applied in order to find the volumes explicitly. Canonical duality between octahedron and hexahedron in the spherical space gives us the possibility to express volume in terms of dihedral angles as well as in terms of lengths of edges. This is a joint work, during A. Mednykh visit to Universidad Tecnica Federico Santa Maria.

1. Introduction

The calculation of volume of polyhedron in hyperbolic and spherical spaces is a very old and difficult problem. The volume of hyperbolic polyhedra in many particular cases was investigated by E. Vinberg [V]. The general formula for volume of tetrahedron remained to be unknown for a long time. Just recently, Y. Choi, H. Kim [ChK], J. Murakami, U. Yano [MY] and A. Ushijima [U] were succeeded in finding such formula. D. Derevnin, A. Mednykh [DM] suggested an elementary integral formula for the volume of hyperbolic tetrahedron. It was discovered by J. Milnor [Ml] and by D. Derevnin, A. Mednykh and M. Pashkevich [DMP] that in the case when all faces of tetrahedron are mutually congruent the volume formula can be obtained in a very explicit way. The volume of the Lambert cube in hyperbolic and spherical spaces was found by R. Kellerhals [K] and D. Derevnin, A. Mednykh [DM], respectively.

In the present report we consider two kinds of spherical polyhedra. The first one is a symmetric octahedron with mutually congruent faces and the second one is an hexahedron (cube) dual to it . We show that

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the elementary volume formulae similar to those for tetrahedron can be obtained.

2. Volume Formula for Symmetric Spherical Octahedron

The symmetric spherical octahedron $\mathcal{O} = \mathcal{O}(a, b, c, A, B, C, \alpha, \beta, \gamma)$ is a spherical polyhedron with eight congruent faces (spherical triangles). Every face on each vertex has angles α, β and γ, sides lengths a, b, and c and between two faces meeting in one side are defined the dihedral angles A, B and C. The notation is as usual, face angle α is the opposite to the side with length a and the dihedral angle A can be found between two faces meeting in a side with length a.

Theorem 1.1 (The Sine-Tangent Rule). *Given a symmetric spherical octahedron $\mathcal{O}(a, b, c, A, B, C, \alpha, \beta, \gamma)$, then the following trigonometric rule holds*

$$\frac{\sin A}{\tan a} = \frac{\sin B}{\tan b} = \frac{\sin C}{\tan c} = T = 2\,\frac{K}{\mathcal{C}}\ ,$$

where K and $\mathcal{C}$ are positive numbers defined by

$$K^2 = (xy - z)(yz - x)(xz - y) \quad \text{and} \quad \mathcal{C} = x^2 + y^2 + z^2 - 2xyz - 1.$$

Here $x = \cos a$, $y = \cos b$ and $z = \cos c$.

Thus, what we have realized is the important fact that a symmetric spherical octahedron is completely determined by its dihedral angles, hence $\mathcal{O} = \mathcal{O}(A, B, C)$. With this identity we are able to prove the following

Theorem 1.2 (Volume of Octahedron). *Given a symmetric spherical octahedron $\mathcal{O} = \mathcal{O}(A, B, C)$ then, we have the following formula for the volume of $\mathcal{O}$*

$$V(\theta) = \int_{\pi/2}^{\theta} \Big(\operatorname{arth}(X\cos\tau) + \operatorname{arth}(Y\cos\tau) + \operatorname{arth}(Z\cos\tau) + \operatorname{arth}(\cos\tau)\Big)\frac{d\tau}{\cos\tau}$$

where $0 \le \theta \le \pi/2$ is a positive number defined by

$$\tan^2\theta + \frac{(1+X)(1+Y)(1+Z)}{1+X+Y+Z} = 0.$$

Here $X = \cos A$, $Y = \cos B$ and $Z = \cos C$.

3. Volume Formula for Symmetric Spherical Hexahedron

Following the same ideas as before, we can obtain similar results for the symmetric spherical hexahedron $\mathcal{H} = \mathcal{H}(a, b, c, A, B, C, \alpha, \beta, \gamma)$, indeed

Theorem 2.1 (The Tangent-Sine Rule). *Given a symmetric spherical hexahedron $\mathcal{H}(a,b,c,A,B,C,\alpha,\beta,\gamma)$, then the following trigonometric rule holds*

$$\frac{\tan A}{\sin a} = \frac{\tan B}{\sin b} = \frac{\tan C}{\sin c} = T = 2\,\frac{\mathcal{C}}{K}\ ,$$

where K and $\mathcal{C}$ are defined by

$$K^2 = -(X+YZ)(Y+XZ)(Z+XY) \quad \text{and} \quad \mathcal{C} = X^2+Y^2+Z^2+2XYZ-1.$$

Here $X = \cos A$, $Y = \cos B$ and $Z = \cos C$.

As before, what is stated in this theorem is that a symmetric spherical hexahedron is completely determined by its dihedral angles, hence $\mathcal{H} = \mathcal{H}(A,B,C)$. In Theorem 1.1 and Theorem 2.1 we easily observe the duality relation between $\mathcal{O}$ and $\mathcal{H}$. Analogously, we obtain:

Theorem 2.2 (Volume of Hexahedron). *Given a symmetric spherical hexahedron $\mathcal{H} = \mathcal{H}(A,B,C)$ then, we have the following formula for the volume of $\mathcal{H}$*

$$V(\theta) = 2\mathrm{Re}\int_{\theta}^{\pi/2}\left(\mathrm{arch}\frac{X}{\cos\tau} + \mathrm{arch}\frac{Y}{\cos\tau} + \mathrm{arch}\frac{Z}{\cos\tau} + \mathrm{arch}\frac{1}{\cos\theta}\right)\frac{d\tau}{\sin\tau}$$

where $0 \le \theta \le \pi/2$ is a positive number defined by

$$\tan^2\theta + \frac{(2XYZ + X^2 + Y^2 + Z^2 - 1)^2}{4(X+YZ)(Y+XZ)(Z+XY)} = 0.$$

Here $X = \cos A$, $Y = \cos B$ and $Z = \cos C$.

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