# On the Milnor alternative for groups of interval diffeomorphisms

Andrés Navas
Universidad de Chile

Given a finitely generated group (provided with a finite and symmetric system of generators), the growth function assigns, to each positive integer $n$, the number of elements of the group that can be written as a product of no more than $n$ generators. One says that the group has polynomial, exponential or intermediate growth, if its growth function has the corresponding asymptotics. (Those notions do not depend on the choice of the finite system of generators.) A deep theorem by M. Gromov establishes that a group has polynomial growth if and only if it is almost nilpotent, *i.e.* if it contains a finite index nilpotent subgroup. Typical examples of groups with exponential growth are those that contain free semi-groups on two generators. The difficult question (raised by J. Milnor) concerning the existence of groups with intermediate growth was answered affirmatively by R. Grigorchuk at the begining of the eighties. Some years later, one of his examples was realized (by R. Grigorchuk himself and A. Maki) as a subgroup of $\mathrm{Homeo}_+([0,1])$. The following result solves by the negative a conjecture of these authors.

**Theorem A.** *The Grigorchuk-Maki's group of intermediate growth has a faithfull action by $C^1$ diffeomorphisms of the interval.*

The method of proof of this theorem allows to give many other examples of subgroups of $\mathrm{Diff}^1_+([0,1])$ with intermediate growth. It relies on the classical construction of Denjoy-Pixton's counter-examples. However, the situation here is more delicate, since in this context it is not possible to improve the regularity to $C^{1+\alpha}$ for any $\alpha > 0$, according to the following result.

**Theorem B.** *For all $\alpha > 0$ every finitely generated subgroup of $\mathrm{Diff}^{1+\alpha}_+([0,1])$ with sub-exponential growth is almost nilpotent.*

This result holds more generally for groups without free semi-groups on two generators. Its proof relies on the rigidity theory for centralizers of diffeomorphisms of the interval. One of the main new issues is an extension of part of that $C^2$ theory to the class $C^{1+\alpha}$. In this direction, Theorem B can be seen as an generalization of the classical Plante-Thurston's Theorem, which says that nilpotent groups of $C^2$ diffeomorphisms of the interval are Abelian.

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**Main reference:** Growth of groups and diffeomorphisms of the interval. Preprint by the author on arxiv.