TRIPLES OF IMAGINARY REFLECTIONS

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Introduction

A (topological) handlebody of genus $g \geq 1$ (respectively, of genus 0) is a compact 3-manifold homeomorphic to the connected sum of g copies of the product to the unit circle with the closed unit disc (respectively, homeomorphic to the closed 3-ball). The interior of a handlebody carries many isometrically different complete hyperbolic structures if $g \geq 1$ (just one for $g = 0$). They also admit many different geometrically finite, complete hyperbolic structure whose injectivity radius is bounded away from zero. Such hyperbolic structures are provided by Schottky groups (we say that the handlebody has a Schottky structure). The space of such Schottky structures on a fixed genus g is the well known Schottky space of rank g.

If M_g is a handlebody with a Schottky structure, its boundary S_g, which is a closed orientable surface of genus g, carries a natural Riemann surface structure (provided by the Schottky group in question). Each isometry $h : M_g \to M_g$ of the handlebody M_g induces a conformal automorphism of S_g if h preserves orientation or an anticonformal automorphism of S_g if h is orientation-reversing. We may think of the isometries of M_g as those conformal/anticonformal automorphisms of S_g which may be continuously extended to the rest of the handlebody.

Let us consider an involutory orientation-reversing isometry τ on the handlebody M_g with a Schottky structure, then it acts naturaly on the boundary Riemann surface S_g as an anticonformal involution of order two. The connected components of the locus of fixed points of τ consists on some number a of isolated fixed points and some number b of compact bordered surfaces. If $b = 0$, then we say that τ is an imaginary reflection; other wise it is called a reflection.

In the case that $b > 0$, that is, when τ is a reflection, Harnack's theorem [5] asserts that the two-dimensional locus of fixed points intersect S_g in a finite number $1 \leq k \leq g+1$ of pairwise disjoint simple loops, called ovals or mirrors. In [8, 9] we have obtained a kind of Harnack's result: $0 \leq a + b \leq g + 1$.

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Let us assume we are given two different involutory orientation-reversing isometries τ_1, τ_2 on the handlebody M_g with a Schottky structure. It is well known [2] that the sum of the ovals of both of them is bounded by $2(g+1)$ and that the bound is sharp. A maximal reflection is one with the maximal number of ovals, that is $g+1$. It is known that a non-hyperelliptic Riemann surface has at most one maximal imaginary reflection. If S_g is hyperelliptic and it has a maximal reflection of S_g, then there are exactly two maximal imaginary reflections, the other is the composition of the hyperelliptic involution with the maximal imaginary reflection. In the case that both τ_1 and τ_2 are imaginary reflections, if we denote by n_j the number of fixed points of τ_j, then the results in [10] asserts that $n_1 + n_2 < 2g/p + 4$, where p is the order of the isometry $\tau_2\tau_1$; moreover, for g even the integer p is necessarily odd. In particular, imaginary reflections with maximal number $g+1$ of fixed points are unique. In [10] it is also observed that for even genus any two imaginary reflections (either in the handlebody case as in Riemann surface case) are conjugated by an orientation-preserving isometry in the handlebody case and by a conformal automorphism in the Riemann surface case. At this point is important to note that there are examples of pairs of imaginary reflections on Riemann surfaces which cannot be extended a isometries for a common handlebody containing the given Riemann surface as conformal boundary [10].

Let us assume we have three different involutory orientation-reversing isometries τ_1, τ_2, τ_3 on the handlebody M_g. In [11] it has been proved that the total sum of ovals of these three involutions is bounded by $2(g+2)$. Results related to the sum of the ovals of any collection of involutory orientation-reversing isometries are given, for instance, in [3, 11, 12, 14]. In this note we consider the case when the three involutions are imaginary reflections. We obtain the following result.

Theorem 1. *Let τ_1, τ_2, τ_3 be three different imaginary reflections on a handlebody M_g of genus $g \geq 2$ with Schottky structure, and denote by n_j the number of fixed points of τ_j. Then,*

(1) *If $g = 2$, then $n_1 + n_2 + n_3 \leq 3$ and the upper bound is attained.*
(2) *If $g = 3$, then $n_1 + n_2 + n_3 \leq 6$ and the upper bound is attained.*
(3) *If $g \geq 4$ is even, then $n_1 + n_2 + n_3 \leq g+5$ and there are infinitely many values of g for which the upper bound is attained.*
(4) *If $g \geq 5$ is odd, then $n_1 + n_2 + n_3 \leq g+7$ and there are infinitely many values of g for which the upper bound is attained.*

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