A Note on the Lucas Circles

Wilfred Reyes

Universidad del Bío- Bío , Chillán

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correo:wreyes@pehuen.chillan.ubiobio.cl

- Given a triangle ABC there is a similar triangle $AB'C'$ such that $B'\,C'$ is the side of an inscribed square. The circumscribed circle to $AB'C'$ is a Lucas Circle. The Three Lucas Circles of a triangle are mutually tangent . This fact was proved in [1] . Here we give an alternative proof and we add some remarks concerning this very interesting system of circles.

  Let ABC be a triangle inscribed in the unit circle Ω in the Euclidean Plane. Following the usual notation throughout we set $\|A\| = 1, \ldots, \|A-B\| = AB = c, \ldots, < AB >= cos2C = 1 - c^2/2, \ldots$.

- For α : $0 \leq \alpha \leq 1$ let O_A denote the circle centred at $(1-\alpha)$. A with radius α . This circle is tangent to Ω at A and it is the image of Ω under the similiraty.

$$X \to X' \ : AX' = \alpha AX, \quad (X' = (1-\alpha)\,A + \alpha\,X)$$

- Also the similarity

$$X \to X'' : \quad AX'' = -\alpha AX, \quad (X'' = (1+\alpha)A - \alpha X), \quad \alpha \geq 1,$$

- Maps Ω onto the circle W_A of radius a that is tangent to Ω at A from the outside. Given β, γ, O_B, O_C, W_B, W_C are defined likewise. Now, O_A and O_B are tangent if and only if

$$\|(1-\alpha)A - (1-\beta)B\| = \alpha + \beta.$$

  Squaring , introducing the stated expression for $< A\,, B >$ and simplifying we get

$$c^2 = 4\alpha\beta/(1-\alpha)(1-\beta),$$

- And a cyclic permutation gives further

$$a^2 = 4\beta\gamma/\,(1-\beta)(1-\gamma), \quad b^2 = 4\alpha\gamma/\,(1-\alpha)(1-\gamma).$$

- The tangency condition applied to the circles W_A, W_B, W_C gives the same formulae but with $+$ instead of $-$.

  It is easy to verify that conversely

$$\alpha = bc/(bc+2a) \quad ,\beta = ca/(ca+2b) \quad ,\gamma = ab/(ab+2c)$$

or , in the case of the outer circles $W_A, \ldots$

$$\alpha = bc/(2a-bc) \quad ,\beta = ca/(2b-ca) \quad ,\gamma = ab/(2c-ab)$$

thus we have characterized the Lucas Circles of a triangle ABC by stating their radii. We proceed now to a geometric explanation. Let $AD = h$ be the height from A in triangle ABC , then from the area-form we have $bc = 2h$, so we may write for the radius of O_A, $\alpha = h/(a+h)$,while the radius of W_A is $\alpha' = h\,/(a-h)$ (we use α' because the parameters are not the same in both cases). Now the respective similaraties imply

$$h' = AD' = \alpha h = h^2/(a+h)$$

$$h'' = AD'' = \alpha' h = h^2/(a-h)$$

So we for the inner and the outer inscribed triangles

$$D'D = h - h' = ha\,/(a+h) = a' = B'\,C',$$

$$D''D = h - h'' = ha\,/(a+h) = a'' = B''\,C''.$$

So $D'D$ is the side of an inscribed square as well as $D''D$,but the vertices B'' and C'' lie on the prolongations of the sides BA and CA . Also we have from the above

$$1/a' \quad + \quad 1/a'' = 2/h,$$

thus the height of the triangle is the harmonic mean of the sides of the outer and the inner inscribed squares.

References

[1] Andreas P. Hatzipolakis , Paul Yiu. The Lucas Circles of a Triangle, Amer. Mat. Monthly ,108 (2001) 444-446.

[2]W. Reyes, The Lucas Circles and the Descartes Formula , Forum Geometricurumn, Vol 3 (2003).